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On the Analytical Basis of Non-Euclidian Geometry.

By W. H. Young, Sc. D., F. R. S.

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§ 1. *Introductory.*

There is, it seems to me, a tendency at the present time to throw dust in the eyes of the mathematical public or rather of the schoolboy public, in respect of the step taken which corresponds to the assumption of Euclid's XIth Axiom. To efforts, now recognized by all mathematicians to be abortive, towards proving the XIth Axiom, have succeeded treatments of the subject matter of Euclid for future engineers and others which gloze over the difficulty this axiom involves. I have recently proposed* to emphasize the empirical character of the axiom by giving to it a form which challenges attention, a form, moreover, having the double advantage that it relates to bounded space, and can be experimentally verified by the dullest schoolboy, even by one to whom the idea of an angle represents an incompletely solved difficulty.

If on the other hand we leave the region of school-books, we find for the most part an air of unreality and half-heartedness in the treatment of the subject. Almost all writers tacitly or expressly assume a knowledge of Euclidian geometry, and express themselves as if the Euclidian hypothesis were the more natural. But further, the possibility of a non-Euclidian system is regarded as demonstrated by means of a correspondence between a selected set of Euclidian entities with the elements of non-Euclidian space. Other writers have made the assumption that Euclidian geometry holds in the smallest parts (im Kleinen).†

The object of the present paper is to show how the refinements of modern analysis enable us to give a perfectly rigid and consistent account of the three geometries which arise when, retaining in all cases Euclid's remaining axioms, we adopt one of the three alternatives which may take the place of Euclid's Fifth Postulate, sometimes called the XIth Axiom. The retained axioms may be expressed in different forms. Adopting Hilbert's classification, they consist of four groups:

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* "On a Form of the Parallel Axiom," 1910, *Quarterly Journal*, Vol. XLI, pp. 353–363.

† Such a method is like that of teaching a foreign language through your own. Non-Euclidian geometry from such a standpoint is relegated to the place of a dead language.

I) The Axioms of Classification;
II) The Axioms of Order;
III) The Axioms of Congruence;
V) The Axioms of Continuity.

Hilbert's fourth group consists precisely of the Axiom of Parallels; that is, the equivalent of Euclid's Fifth Postulate.[*] The form of the three alternative axioms, characterizing the three geometries under discussion, referred to above in the first paragraph, is as follows:

Let it be assumed that there is one triangle in the bounded convex space we are considering, such that the length of the line joining the middle points of the two sides is

1) *less than (Lobatchefsky-Bolai geometry),*
2) *equal to (Euclidian geometry),*
3) *greater than (Riemann geometry)*

half the third side.[†] It then follows that, in the respective cases, the sum of the angles of the triangle is less than, equal to, or greater than two right angles;[‡] and conversely, if this holds, the corresponding statement in the axiom is true. Also the same is true of every triangle.[§]

Moreover, the defect of a variable triangle QAP, that is the difference between the sum of its angles and two right angles, has zero as limit, if the point P moves along a straight line towards A as limiting point.[||] Hence, it easily follows that, *if we have a triangle of varying shape and position, which is in course of shrinking up to a point, its defect has zero as limit,* since[¶] the defect of a triangle is less than that of another triangle containing it.

---

[*] D. Hilbert: "Grundlagen der Geometrie," Leipzig, Teubner, 2nd Edition, 1903. I should like here to refer to Enriques' collection of essays by various Italian authors, "Quistioni Riguardanti la Geometria Elementare," Bologna, 1900; the second volume of the German translation is now in the press. There are various accounts of such sets of axioms in English; *e. g.*, in Halsted's "Rational Geometry." See also A. N. Whitehead's two contributions to the Cambridge Tracts in Mathematics.

[†] *Loc. cit.*, § 2.

[‡] *Loc. cit.*, § 15.

[§] *Loc. cit.*, § 16.

[||] *Loc. cit.*, §§ 13, 14.

[¶] The classical method of proving this statement is to deduce it in steps, at each step comparing two triangles with two common vertices, say B and C, and the remaining vertices in a straight line with C, say ADC. If then, in the Lobatchefsky geometry,

$$\pi - ABC - BAC - CAB < e,$$

we have at once,

$$(\pi - ABD - ADB - BAD) + (\pi - DBC - DCB - BDC) < e.$$

Since each of the expressions in brackets is positive, each must be $< e$. Similarly, the statement may be proved in the other geometries.

The line of argument which I have adopted is one of those indicated by Killing in a couple of pages of his "Grundlagen der Geometrie." The best justification for the detailed account here given will be found in the comparison, made by the reader himself, between the paper before him and these pages of the "Grundlagen der Geometrie." One other writer appears to have treated the subject from a similar point of view, Girard in his Thèse. The method he employs is, however, quite different, and, as it appears to me, less natural from the point of view of analysis. The definition of the sine, cosine and tangent as limits presents itself inevitably to the modern analyst, whereas this can hardly be said of the definitions given by Girard.

On the basis here given the whole edifice of non-Euclidian or Euclidian geometry can be securely built. The discussion terminates as soon as we have obtained the formulæ connecting the sides and angles of a triangle. For the convenience of readers the inequalities are stated for the Lobatchefsky case. The discussion here given is, however, perfectly general. It is, moreover, easy to deduce all the results in three dimensions from the plane formulæ here obtained, ranging from the theorem, sometimes tacitly assumed without justification, that there is in all three geometries such a thing as the angle between two planes, to all the interesting properties in Clifford's Theory of Parallels in Elliptic Space, and of the Grenzfläche in the Lobatchefsky-Bolai geometry.

§ 2. *The Tri-Rectangle.*

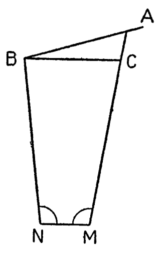


FIG. 1.

In the Lobatchefsky geometry the sum of the angles of any triangle is less than two right angles, and therefore that of the angles of a quadrilateral is less than four right angles. Hence, if the quadrilateral has a pair of opposite sides BN, CM (Fig. 1) equal, and the angles at N and M right angles, the other angles, being by symmetry equal, are both acute. It follows that the perpendicular from B to BN falls outside the quadrilateral. Hence, if from any point A on one of two straight lines AM, BN, perpendicular at N and M to the same straight line, we drop a perpendicular AB on to the other line BN to form a quadrilateral $ABNM$ with three right angles, the side BN is less than AM. That is, *in a quadrilateral with three right angles, a side between two right angles is less than the opposite side.*

§§ 3–9. *The Sine-Ratio and the Sine.*

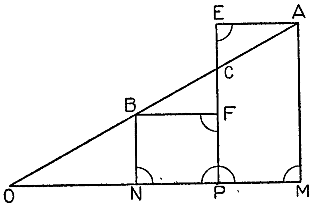


FIG. 2.

§ 3. Take any acute angle, let O be its vertex, and from A and B, any two points on one of the arms, draw AM, BN, perpendicular to the other arm. Let C be the middle point of AB, so that

$$2\,OC = OA + OB.$$

Draw CP perpendicular to the other arm, and AE perpendicular to CP. On the other side of C from E on the straight line CP take F, so that

$$CF = CE.$$

Then the triangles CFB, CEA are congruent, since they have equal angles at C and the containing pair of sides of the one equal to that of the other. Hence, the angle at F, like that at E, is a right angle.

From the result at the end of the preceding article it follows that

$$EP < AM,$$
$$FP < BN,$$

and therefore, since the sum of FP and EP is twice CP,

$$2\,CP < AM + BN.$$
$$\frac{CP}{OC} < \frac{AM + BN}{OA + OB}.$$

Now let us denote the ratio of AM to AO by a, that of BN to OB by b, and that of CP to OC by c. Then the last inequality becomes

$$c < \frac{a\,.\,OA + b\,.\,OB}{OA + OB} \qquad (1),$$

whence, if we know from other sources that

$$b < c,$$

it follows that

$$c < a.$$

§ 4. Now let us reflect Fig. 2 in the line OM, and denote the reflections by the same letters as before, distinguishing by a dash. Then, if B is the middle point of OC, B' is the middle point of OC', so that (§ 1)

$$BB' < \tfrac{1}{2}\, CC',$$

whence, since the line at which we reflect bisects both BB' and CC',

$$BN < \tfrac{1}{2}\, CP.$$

Hence, since

$$OB = \tfrac{1}{2}\, OC,$$

$$b < c.$$

§ 5. From the last two articles it follows that, if

$$OB : OC : OA = 1 : 2 : 3,$$

$$b < c < a.$$

Hence, it follows by induction that, *if OB, OC and OA are in the ratio of any three consecutive integers, b, c and a are in ascending order of magnitude.* For, suppose this to have been proved for the three consecutive integers $(n - 1)$, n, $(n + 1)$, and let

$$OB : OC : OA = n : n + 1 : n + 2.$$

Then we know that

$$b < c;$$

therefore, by § 3,

$$c < a,$$

which proves the statement for the next trio of consecutive integers. But it has been proved for the three successive integers 1, 2, 3; hence the induction is complete, and it is true for any three consecutive integers.

§ 6. Hence, if OA be divided into any number of equal parts, the ratio of the perpendicular from any point on OA to its distance from O, or as we shall say, *the sine-ratio*, increases monotonely from point to point of division passing from O to A. Since this is true at each stage when we divide successively into 2, 4, 8, ... parts, it follows that, if x is the sine-ratio at any point of binary division X, and y the sine-ratio for another of these points Y, then if X lies between O and Y,

$$x < y.$$

§ 7. The binary points of division, though dense everywhere,* are only countable. In order to prove further that the statement at the end of the preceding article is true when X and Y are not neccessarily points of binary division, we have to show that x is a continuous function of the position of the point X.

---

* For the explanation of this and other terms borrowed from the Mengenlehre, the reader may refer to "The Theory of Sets of Points," by W. H. Young and Grace Chisholm Young, Cambridge University Press (1906).

Now since the angles of a triangle are together less than two right angles, the angle OAM is acute. Also, since (§ 1) the defect from two right angles of the sum of the angles of a triangle has zero for limit when one of the vertices moves along a straight line up to the other as limiting point, it follows that the sum of the angles of the quadrilateral $AMPE$ of Fig. 2 is as near as we please to four right angles, provided A is near enough to C. Hence, we may assume that A is sufficiently near C for the angle at A, the only angle of the quadrilateral which is not a right angle, to be greater than the acute angle OAM.

This being so, E falls on the other side of C from P, and therefore

$$CP < PE < AM,$$

using the result of § 2. But

$$AM < AC + CP + PM;$$

thus

$$CP < AM < CP + AC + PM.$$

Now when A moves up to C as limit, M moves up to P, both moving on straight lines, viz., the arms of the angle. Hence, AC and PM each have the unique limit zero. Thus the extremes in the last inequality have the same limit CP, so that AM has also CP as unique limit.

Since also OA has OC as unique limit, the ratio AM/OA has as limit the ratio CP/OC.

This proves that the *sine-ratio is a continuous function.*

§ 8. Now *a continuous function which is monotone with respect to an everywhere dense set of points is a monotone function.* For let X lie between O and Y,

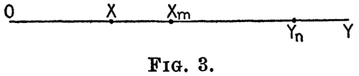


FIG. 3.

and let Y_n be any point of the everywhere dense set between X and Y, and X_m any point of the same set between X and Y_n. Then denoting the values of the function at these points by x, x_m, y_n and y, and supposing the function to *decrease* as we move from one point to another of the everywhere dense set in the direction towards O,

$$x_m < y_n.$$

Let X_m move along a sequence of the points of the everywhere dense set in the direction of O and have X as limiting point. Then the quantities x_m diminish down to their unique limit, which is x, since the function is continuous. Hence,

$$x < x_m .$$

Similarly, if Y_n moves towards Y as limit along points of the same set,

$$y_n < y.$$

Hence,

$$x < x_m < y_n < y.$$

This shows that the function is a monotone function.

§ 9. We have now shown that the sine-ratio is a continuous function which is monotone decreasing as the point X approaches O. Hence, this ratio has a unique limit at the point O, and this limit is less than any value of the ratio. This limit we shall call *the sine of the angle AOM*, and write in the usual way sin AOM.

§§ 10–12. *The Cosine-Ratio and the Cosine.*

§ 10. Let AM and BN be any two perpendiculars from one of the arms ABO of an acute angle at O on to the other arm, B being nearer to O than A. Bisect NM at P, and draw PC perpendicular to NM, meeting OA (as of course it must do, since it can not meet AM nor BN) at C.

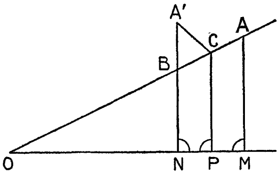


FIG. 4.

Then, since the sum of the angles of a quadrilateral is less than four right angles, the angle OBN is greater than the angle OAM. Hence, if A' is the reflection of A in CP, the angle $BA'C$, being equal to OAM, is less than the angle $A'BC$, that is OBN. Hence,

$$BC < CA',$$

that is,

$$BC < CA.$$

§ 11. If the points B and N coincide with the point O itself, the argument of the preceding article still holds. Hence, if we divide OM into any number of equal parts and construct the perpendiculars to OM at the points of division, these divide OA in such a way that the stretch between two consecutive points of division increases monotonely as we recede from O.

Denoting the points of division on OM by $P_1, P_2, \ldots, P_{n-1}$, and those on OA by the letter Q with the same indices, we have

$$r \,.\, OQ_{r-1} = (r-1)\, OQ_{r-1} + OQ_{r-1} = (r-1)\, OQ_r - (r-1)\, Q_r Q_{r-1} + OQ_{r-1}.$$

But

$$OQ_{r-1} < (r-1)\, Q_r\, Q_{r-1},$$

since it consists of $(r-1)$ parts each less than $Q_r\, Q_{r-1}$. Hence,

$$r \,.\, OQ_{r-1} < (r-1)\, OQ_r,$$

so that

$$OQ_{r-1}/(r-1) < OQ_r/r,$$

whence

$$OP_{r-1}/\,OQ_{r-1} > OP_r/\,OQ_r.$$

Calling the ratio here considered — viz., that of the base OP_r to the hypotenuse OQ_r of the right-angled triangle OP_rQ_r — *the cosine-ratio* at the point P_r, this shows that, if OA be divided into any number of equal parts, *the cosine-ratio is a function which, as we move along from point to point of division towards the vertex O, increases monotonely.*

§ 12. Hence, if we divide OA by continued bisection, and so get the set of points of binary division, the cosine-ratio at the points of binary division is a function which decreases monotonely as we approach O.

Hence, by § 8, if this ratio is a continuous function of the point, it is a monotone increasing function as the point approaches the vertex O. Now it is evident that the cosine-ratio is continuous, for if C be any point between B and A, and P be the foot of the perpendicular from C on OM (Fig. 4), this perpendicular does not meet AM nor BN, since two angles of a triangle are together less than two right angles; therefore it meets the stretch NM, so that P lies between M and N. Hence, if M moves up to N as limiting point, and therefore certainly passes P, A must pass C. Thus A moves up to B as limiting point, since it can not have as limiting point any point C on the other side of B from O. Thus not only OM has ON for limit, but OA has OB, and therefore $OM/\,OA$ has $ON/\,OB$ for limit. *This proves the continuity, and therefore also the monotony of the function constituted by the cosine-ratio $OM/\,OA$, as M moves towards O.*

§ 12. Since the cosine-ratio increases monotonely as the point M approaches O, it has a unique limit at the point O, which is at the same time its upper bound. This limit we shall call *the cosine of the angle* AOM, and write in the usual way $\cos AOM$.

§ 13. *The Tangent, Cotangent, Secant and Cosecant.*

We have seen that the sine-ratio decreases and the cosine-ratio increases; hence it follows that their ratio, which is the limiting value of the ratio of the perpendicular AM to the base OM, decreases, and has therefore a unique limit at O, which is at the same time its lower bound. This limit is called *the tangent of the angle* AOM, and is denoted by $\tan AOM$.

The reciprocals of the sine, cosine and tangent are called *the cosecant, secant* and *cotangent*, or $\operatorname{cosec} AOM$, $\sec AOM$, $\tan AOM$, respectively.

§ 14. *Bounds of the Sine and Cosine.*

Since the angles of a triangle are less than two right angles, so that in a right-angled triangle the right angle is the greatest angle, and therefore the hypotenuse the greatest side, it follows that the ratios whose limits are respectively the sine and cosine are both less than unity, and therefore *both the sine and the cosine of an acute angle lie between* 0 *and* 1, *both inclusive.* Again, since the base is less than the hypotenuse, the ratio of the perpendicular to the hypotenuse is less than the ratio of the perpendicular to the base. Hence, *the sine is less than or equal to the tangent.*

§ 15. *Monotony of the Sine, Cosine and Tangent.*

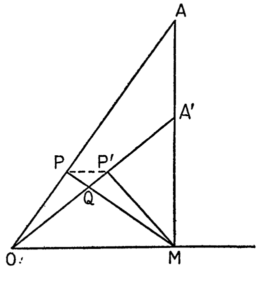


FIG. 5.

Let AOM be any acute angle, and AM perpendicular to OM. Then, if $A'OM$ is any smaller angle, the ray OA' falls inside the angle AOM, and therefore cuts AM, say at A'. Therefore,

$$AM/OM > A'M/OM.$$

Since this is true for every position of A as it moves up to O as limiting point, we get, proceeding to the limit,

$$\tan AOM \geq \tan A'OM.$$

Thus *the tangent increases as the acute angle increases.*

Again, since the angle at M is a right angle, the angle $AA'O$ is obtuse. Hence,

$$OA > OA',$$

so that

$$OM/OA < OM/OA',$$

whence, as before, proceeding to the limit,

$$\cos AOM \leq \cos A'OM.$$

Thus *the cosine decreases as the angle increases.*

Again, if MP be perpendicular to OA, P falls between O and A, since otherwise one of the angles made by MP with OA would be less than AOM or than OAM (both of which are acute angles), since the exterior angle of a triangle is greater than the interior opposite angle.

Since P lies between O and A, MP meets OA' inside the triangle OAM, say at Q. Then $MP > MQ >$ the perpendicular MP' from M on OA'. Hence,

$$MP/OM > MP'/OM.$$

Since this is true for each position of M, as M moves up to O as limiting point, we get, proceeding to the limit,

$$\sin AOM \overset{-}{>} \sin A'OM.$$

This proves that *the sine of an acute angle increases as the angle increases.*

Thus, as the angle increases, the sine, cosine and tangent are monotone functions of the angle, the sine and tangent increasing and the cosine decreasing. (N.B. It will be shown later that the alternative sign of equality in the inequalities proving the monotony of the sine, cosine and tangent is inadmissible.)

§ 16. *Values of the Sine, Cosine and Tangent of any Acute Angle.*

It follows from the definitions of the sine, cosine and tangent, as the unique limits of certain positive ratios, that they are never negative. It will now be shown that, the angle being as hitherto an acute one, *the sine, cosine and tangent are themselves positive, and that the sine and tangent have zero as unique limit, and the cosine has unity as unique limit, when the angle decreases indefinitely.*

Divide a right angle by continued bisection into n equal parts, and divide an adjacent right angle in the same way (Fig. 6). Let $OK_0 = OK_{2n}$ be equal lengths on each side of the common arm OK_n of the two right angles, and take OK_2, OK_4, ... on the alternate arms of the parts each equal to OK_0. Then, if we join K_0K_2, K_2K_4, ... all round to K_{2n}, we get a polygon whose last side is $K_{2n}K_0$, and whose n other sides are each equal to K_0K_2. Hence,

$$n\,.\,K_0K_2 > K_{2n}K_0 > 2\,OK_0.$$

But, if K_0K_2 meet the intermediate dividing line in K_1, so that the angle K_0OK_1 is the first of the n equal parts into which we had divided the first right angle, the angles OK_1K_0 and its adjacent angle OK_1K_2 are equal, and each is a right angle. Also $K_0K_1 = K_1K_2$, all this following from the congruence of the triangles OK_1K_0 and OK_1K_2. Hence,

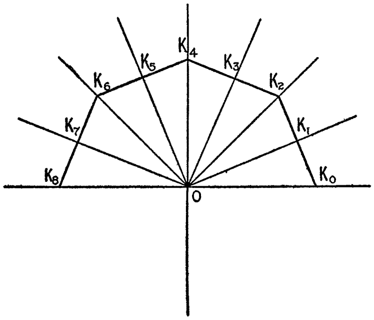


FIG. 6.

$$2n\,.\,K_0K_1 > 2\,OK_0,$$

so that

$$K_0K_1\,/\,OK_0 > 1/n.$$

Since this is true for every position of the point K_0 as it moves up to O as limiting point, we get, proceeding to the limit,

$$\sin K_0OK_1 \geq 1/n.$$

Now whatever acute angle be given, we can find n so large that the given angle is greater than the angle K_0OK_1, which is the n-th part of a right angle. Since the sine increases with the angle, it follows that the sine of the given angle is $\geq 1/n$, and is therefore positive.

Since the tangent of an acute angle is greater than the sine, this proves that both the sine and the tangent of an acute angle or positive, and therefore that the cosine, which is the quotient of the sine by the tangent, (since neither is zero) is also positive.

§§ 17, 18. *Limiting Values of the Sine, Cosine and Tangent.*

§ 17. To find the unique limits approached by the sine, cosine and tangent when the angle diminishes without limit, let us return to Fig. 5. Let OM be any stretch and AM a line perpendicular to OM and of one n-th of the length of OM. Then

$$0 < \sin AOM < \tan AOM < AM/OM < 1/n,$$

and, since $OA < OM + MA$,

$$1 < OM/OA + MA/OA < \cos AOM + 1/n.$$

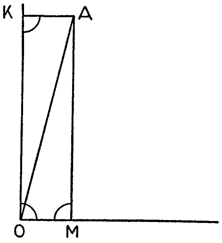


FIG. 7.

Increasing n without limit, so that the angle AOM diminishes without limit, these inequalities show that *the sine and tangent have, when the angle decreases indefinitely, the limit zero and the cosine the limit unity.* Since the sine, cosine and tangent are monotone, these limits are, of course, unique.

§ 18. We can now find the unique limits of the sine, cosine and tangent when the acute angle increases with a right angle as limit. For, let AOM be an acute angle (Fig. 7), AM perpendicular to OM, and let MOK be a right angle, and AK perpendicular to OK. Then (§ 2)

$$OK < AM, \text{ and } OM < AK.$$

Hence,

$$OK/OA < AM/OA \text{ and } OM/OA < AK/OA.$$

Letting the point A move along the arm AO of the acute angle up to O as limiting point, we get, proceeding to the limit,*

$$\cos AOK \leq \sin AOM \text{ and } \cos AOM \leq \sin AOK.$$

Using the results of the preceding article, we see that, *as the angle AOM increases up to a right angle, its cosine has the limit unity and its sine the limit zero; hence, also, its tangent has the limit $+\infty$.*

---

* See § 22, where it is shown that the sign of equality must be taken.

§ 19. *Sine, Cosine and Tangent of a Zero Angle and of a Right Angle.*

So far we have only defined the sine, cosine and tangent of an acute angle. We now extend our definition by assigning to these functions of the angle as values when the angle is zero or a right angle the values of their unique limits. Thus

$$\text{sine } 0 = 0, \quad \text{cosine } 0 = 1, \quad \text{tangent } 0 = 0,$$

$$\textit{sine of a right angle} = 1, \quad \textit{cosine of a right angle} = 0,$$

$$\textit{tangent of a right angle} = +\infty .$$

With these definitions the sine, cosine and tangent are monotone from the value zero to the value a right angle, and they are all continuous for the extreme values of the angle. For the value a right angle, the tangent is only continuous in the extended sense, since it has the value $= +\infty$ there.

§ 20. *Continuity of the Cosine.*

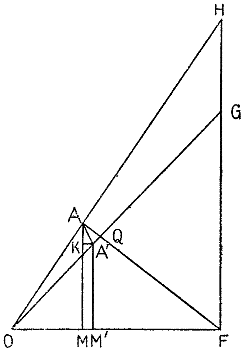


FIG. 8.

Let HOF be any acute angle and GOF any smaller acute angle inside HOF. From any point F on the common arm of the two angles draw FA

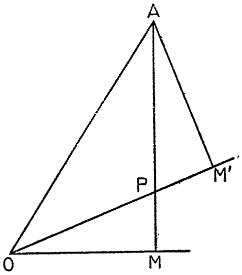


FIG. 9.

perpendicular to OH. Let FA meet OG in Q. Then, since in a right-angled triangle the hypotenuse is the greatest side, $OA < OQ$. Cut off from OQ $OA' = OA$, and draw AM and $A'M'$ perpendicular to OF.

If we rotate the ray OA' round till A' coincides with A, the angle $A'OM'$ being less than the angle AOM, OM' (Fig. 9) will cut AM, at P say. The angle APO will then be obtuse, since it is the exterior angle of a triangle PMO having a right angle at M. Hence, the point M' cannot lie on the same side of P as O, or the angles of the triangle APM' would have a sum greater than two right angles. Thus P lies between O and M'. Hence,

$$OM < OP < OM'.$$

Also (Fig. 8), since the angles OAM and $OA'M'$ are acute, and the angle OAF is a right angle and $OA'F$ greater than the obtuse angle OQF, it follows that M and M' both lie between O and F, so that the order of these points is $OMM'F$. Hence,

$$0 < \frac{OM'}{OA'} - \frac{OM}{OA} = \frac{MM'}{OA}. \tag{1}$$

Now draw $A'K$ perpendicular to AM. Then since the hypotenuse is the greatest side of a right-angled triangle,

$$A'K < A'A,$$

and since the equal angles OAA' and $OA'A$ are both acute, so that the angle $AA'Q$ is obtuse,

$$AA' < AQ.$$

Also, since the quadrilateral $A'M'MK$ has three right angles, we have, by § 2,

$$MM' < A'K.$$

Hence finally,

$$MM' < AQ.$$

Using (1), we thus get

$$0 < \frac{OM'}{OA'} - \frac{OM}{OA} < \frac{AQ}{OA}.$$

Letting the point F move along its ray OF to O as limiting point, the point A also moves up to O as limiting point, since, if L is any point on OH, the angle FLO becomes acute as soon as F has passed the foot of the perpendicular from L on OF, and therefore, when this is the case, A lies between O and L. Thus the whole figure $FQAA'KMM'$ shrinks up to the point O. Hence, we get, proceeding to the limit with the preceding inequality,

$$0 < \cos GOF - \cos HOF < \tan HOG.$$

Now when the angle HOG approaches zero, either by the ray OH turning round till it coincides with OG, or by OG turning round till it coincides with OH, tan HOG has, as has been shown in § 17, the unique limit zero. Thus cos GOF — cos HOF has the unique limit zero. In other words, *the cosine is a continuous function of the angle.*

§ 21. *Uniform Convergence of the Cosine-Ratio to the Cosine.*

Since the cosine-ratio converges monotonely to the cosine, and the cosine is a continuous function of the angle, it follows that *the cosine-ratio converges uniformly to the cosine.** In other words, if AOM is a right-angled triangle and the ray OA turns round to a definite limiting position OB, while at the same time the point A moves up to O as limit, then OM/OA converges uniformly to cos BOM as unique limit.

The cosine-ratio is here considered as being a function of two variables; first the angle AOM, or say z, and secondly the length OM, or say x. Let this function be $f(x, z)$, and let the cosine be $F(z)$. Then

$$F(z) = \underset{x = 0}{\text{limit}} \, f(x, z),$$

by definition; but the equation

$$F(z_0) = \underset{x = 0,\; z = z_0}{\text{limit}} \, f(x, z)$$

expresses the uniform convergence of the cosine-ratio. In the first equation z is constant, and the limit is a single limit; in the second both x and z vary and the limit is a double limit.

Again, if AOM is a triangle of changing form and position, having a right angle at M and a hypotenuse AO which shrinks up with zero as unique limit, and if the angle AOM has a definite limit, say a, then OM/OA converges uniformly to cos a. For we only have to redraw the changing triangle AOM in a new figure in which the point O and the straight line OM are fixed, and this result follows at once from the preceding.

We shall apply this result to prove that the sign of equality must be taken in the relations found in § 18.

---

* By a well-known theorem.

§ 22. *Sine and Cosine of Complementary Angles.*

In the figure of § 17, denoting the angle AOM by x and the angle KOA by y, so that $x+y$ is a right angle, the angle OAM is less than y, but approaches y as limit as A moves up to O along the fixed ray OA. Hence, by the preceding article, AM/OA converges uniformly to $\cos y$ as unique limit.

But, by definition, AM/OA converges to $\sin x$ as unique limit. Hence,

$$\sin x = \cos y.$$

In words, the sine of an angle is the cosine of its complement, and the cosine of an angle is the sine of its complement.

§ 23. *Continuity of the Sine and Uniform Convergence of the Sine-Ratio.*

Now if x changes with a unique limit x_0, its complement y changes with y_0 as unique limit, where y_0 is the complement of x_0. Hence, since $\cos y$ approaches $\cos y_0$ as unique limit, it follows from the preceding that $\sin x$ approaches $\sin x_0$ as unique limit. Thus *the sine is a continuous function of the angle.*

Hence, *since the sine-ratio converges monotonely to the sine, it converges to it uniformly.*

§ 24. *Continuity in the Extended Sense, of the Tangent, Cotangent, Secant and Cosecant, and Uniformity of Approach of the Ratios to their Limits.*

Hence, it follows that the tangent, cotangent, secant and cosecant, being the ratios of continuous non-zero functions as long as we are dealing with acute angles, are continuous functions, and therefore the convergence of the different ratios defining these functions of the angle to their limits is uniform. This is still true for the extreme values zero and a right angle, the continuity being in the extended sense, when the value $+\infty$ is allowed, and the continuity being on one side only, viz. for approach to zero from above and to a right angle from below.

§ 25, 26. *Extension to Angles of Any Magnitude.*

§ 25. With the usual convention as to signs, we now define the sine and cosine of any angle to be the unique limits approached by the sine- and cosine-ratios, viz. the ratios of the perpendicular and base respectively to the hypotenuse of the right-angled triangle AOM, got by drawing AM from a point on one arm of the angle on to the other.

That these limits are unique follows at once from the fact that in every case these ratios are numerically the same as those derived from a certain acute angle. We have in fact the usual relations,

$$\begin{array}{ll} \sin(-x) = -\sin x, & \cos(-x) = \cos x, \\ \sin(180^\circ - x) = \sin x, & \cos(180^\circ - x) = -\cos x, \\ \sin(180^\circ + x) = -\sin x, & \cos(180^\circ + x) = -\cos x. \end{array}$$

Also the addition or subtraction of any number of multiples of four right angles leaves the sine and cosine unaltered.

Here 180° is used for two right angles.

§ 26. With this convention as to the meaning of the terms used, it follows immediately from what has been proved that *the sine and cosine are continuous functions of the angle,* their values for any multiple of a right angle being defined as the values of their unique limits there, and that they lie between 0 and 1 inclusive. They are monotone between any two adjacent multiples of a right angle. The relation

$$\sin x = \cos y$$

holds for all values of x and y such that $x + y$ is a right angle. We have in fact the same picture as in Euclidian trigonometry.

§§ 27–37. *The Circle and Circular Measure.*

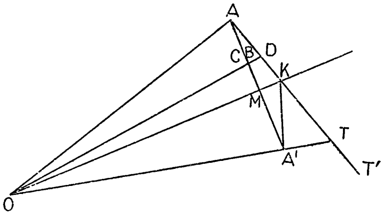


FIG. 10.

§ 27. *The Circle. A circle* is the locus of a point whose distance from a fixed point O, called *the center* of the circle, is constant, say r, and is called *the radius* of the circle.

Whatever bounded space we are working in, if O is an internal point of that space, there is a certain value r_0 such that for all smaller values of r, there is a point of the circle on every half-ray through O. We shall assume that r has such a sufficiently small value.

Let A be any point of the circle and AT' perpendicular to OA and of length less than one n-th of the radius r. Then if the angle AOA' is less than the angle AOT', so that OA' meets the stretch AT', say at T, $OT > OA$, so that the circle cuts the stretch OT, say at A'.

Thus every point of the straight line AT', except A, is *outside* the circle; that is, is at a distance from O *greater* than the radius r.

Let OCD be any ray inside the angle AOA', meeting AA' in C and AT in D. Then, as we saw, $OD > r$, and, since the angle $OCA > OA'A$, that is $OCA > OAA'$, it follows that $OC < OA$. Hence, the point B, in which OD meets the circle, lies between C and D.

Now the angles OAA' and $OA'A$ being equal are acute; therefore, the angle $AA'T$ is obtuse. Hence,

$$AA' < AT < AT' < r/n.$$

Also since in like manner the angles ABO, $A'BO$ are acute,

$$AB < AD \text{ and } A'B < A'D;$$

so that

$$AB + A'B < AD + A'D < AD + DT + A'T < 2\,AT < 2\,r/n.$$

§ 28. Let us now inscribe in the circle a polygon, and let p_n be its perimeter, also let the angles subtended by its sides at the center be each less than the angle AOT', above used. Also let us take another polygon, of perimeter p_{n+1}, having among its vertices all the vertices of the former polygon,

$$p_n < p_{n+1}.$$

Now each of the vertices of the latter polygon adds, as we saw, at most $2\,r/n$ to the perimeter; hence, if m be the number of new vertices of the second polygon,

$$p_{n+1} - p_n < 2\,m\,r/n.$$

§ 29. Thus if we inscribe a series of polygons in the circle in the manner indicated in the preceding article, by adding at each stage new vertices, the perimeters of the successive polygons will form a monotone increasing sequence of numbers, and have therefore a unique limit, viz., their upper bound. Let this be done in such a way that the integer n, used in determining the angle AOT' of § 27, increases by 1 at each stage, so that the angle subtended at the center by any side of the n-th polygon is less than the angle AOT', where $AT' < r/n$.

§§ 30–36. *The Length of the Circle.*

§ 30. Let I by the upper bound of the perimeters of all possible polygons inscribed in the circle of radius r. Then assigning any positive quantity e, we can find an inscribed polygon whose perimeter p' is greater than $I-e$. Let this polygon have m vertices; then we can choose n so large that

$$n > m\,r/e.$$

Let us form a new polygon, of perimeter p_n'', having as vertices all these of the polygon of perimeter p' and also all those of the n-th polygon described as in the preceding article of perimeter p_n.

Then p_n'' is greater than either p' or p_n. But since the number of new vertices inserted in the polygon of perimeter p_n is at most m, it follows from § 28 that $p_n''-p_n$ is less than $2\,m\,r/n$; that is, $< 2e$. Thus

$$I-e<p_n''<p_n+2e.$$

Since this is true for all greater values of n, the limit of the p_n's $\overline{>} I-3e$. This being true for all values of e, the same limit $\geq I$. But the sign $>$ is inadmissible, since I is the upper bound of all possible perimeters. Thus the limit of the p_n's is I.

§ 31. We shall next show that AT is the tangent to the circle, so that *the tangent is the line perpendicular to the radius through the point in which the latter meets the circle*, which is the point of contact of the tangent. For though the angle OAA' is acute, it has a right angle as limit, when the point A' moves along the circle up to A as limit, since the limit of the sum of the two equal angles OAA' and $OA'A$ with AOA', which decreases indefinitely, is two right angles. Hence, whatever line be drawn inside the right angle OAT', there will be a position of A' such that AA' lies inside the angle made by that line with AT'. Thus that line lies partially inside the circle. The same is true, of course, on the other side of OA. Hence AT is the only straight line through A which lies entirely outside the circle, with the exception of the single point A, which lies on the circle. Thus AT is the tangent to the circle.

§ 32. Hence, it easily follows that, if M be the middle point of AA', and OM meet AT in K, $A'K$ is the tangent at A', and $KA=KA'$. For the triangles AMO, $A'MO$ are congruent, since their sides are equal; therefore the point A' is the reflection of A in OK, whence the equality of the lengths and the angles in the two figures $AOMK$, $A'OMK$ follows.

§ 33. Now as A' moves along the circle up to A as limiting point, the angle OAA' has OAT' as limit, and therefore the angle KAM has zero as limit. Hence, by § 21, AM/AK converges uniformly to the cosine of zero, that is to unity, as limit. Hence,

$$AA'/(AK + KA')$$

is less than unity, but has unity as limit, when the angle AOA' is indefinitely decreased.

§ 34. Let us now circumscribe a polygon about the circle, and let p be its perimeter. Joining the points of contact, we get an inscribed polygon; let its perimeter be p'. We may suppose AA' to be one side of the latter polygon, AK and KA' the corresponding sides of the circumscribed polygon (Fig. 10). It then follows from the preceding article that p'/p is less than unity, but has unity as limit, when the angles between the radii to the points of contact of the circumscribed polygon are indefinitely diminished.

Hence the quantity I, previously defined as the upper bound of the perimeters of inscribed polygons, is the unique limit of the perimeter of a sequence of circumscribed polygons, when the angles between the radii to the consecutive points of contact diminish indefinitely, or, which is the same thing, when the points of contact form a set dense everywhere on the circle.

§ 35. It is easily seen that *I is the lower bound of the perimeters of all circumscribed polygons.* For if AK and KA' are consecutive sides of any circumscribed polygon, we obviously get a poylgon of less perimeter by inserting M, the middle point of the arc AA', as a new vertex. Thus we get a monotone decreasing sequence of perimeters, by continually bisecting the arcs between the points of contact and adding the points of bisection as new vertices. Since the limit of the perimeters is then I, it follows that the perimeter of the polygon with which we started is greater than I, so that I is the lower bound of the perimeters of all circumscribed polygons.

§ 36. This being proved, the quantity I is defined to be the *length of the circle.* Now if the angle AOK is fixed, and we diminish the radius of the circle indefinitely, the ratio AK/OA diminishes, and has as limit $\tan AOK$. Hence if p is the perimeter of the circumscribed polygon whose points of contact lie on fixed rays through the common center of the decreasing circles, p/r diminishes. Since this is true for all sets of fixed rays, it is true when we replace p by its lower bound. Thus I/r diminishes, and has therefore a unique limit when r is indefinitely diminished; viz., its lower bound. *This limit we denote by* 2π.

§ 37. If the angle AOK is one n-th of two right angles, it follows from the preceding article that, denoting this angle by a_n, $2n \tan a_n$ diminishes monotonely as n increases and has 2π as limit.

Similarly if, instead of taking a complete inscribed polygon, we take that part of it which is bounded by radii enclosing any fixed angle, we see that the upper bound of the length of the arc of an inscribed polygon and the lower bound of the length of the arc of a circumscribed polygon are equal, and that the common value is the unique limit of the length of the arc either of an inscribed or of a circumscribed polygon, when the points of contact of the latter or the vertices of the former become dense everywhere. This unique limit is defined to be *the length of the arc of the circle* with which we started. This definition is allowable, since the length, so defined, clearly has the property that *the length of the sum of two adjacent arcs is the sum of the lengths of those two arcs*, and also the property that *if an arc is the unique limit of a sequence of arcs, its length is the limit of their lengths.*

It then follows, by the same argument as before, that, *when the angle subtended at the center by the arc under consideration is kept unaltered, and the radius of the circle is diminished indefinitely, the ratio of the length of the arc to the radius diminishes, and has therefore a unique limit.*

Calling this limit θ, it follows by the same argument as before, that, if the angle subtended at the center by the arc of the circle considered be m/p of two right angles, θ is the unique limit of $mn \tan a_{np}$. Hence,

$$\frac{\theta}{\pi} = \frac{\underset{n=\infty}{\text{limit}}\; mn \tan a_{np}}{\underset{n=\infty}{\text{limit}}\; np \tan a_{np}} = \frac{m}{n} = \textit{ratio of the angle to two right angles.}$$

Again, if the ratio of the angle to two right angles is irrational, we can, corresponding to each value of the integer p, assign an integer m, such that the angle lies between $m a_p$ and $(m+1)\, a_p$. Hence, by the same argument,

$$\frac{m}{p} < \frac{\theta}{\pi} < \frac{m+1}{p}$$

for each value of p, so that,

$$\frac{\theta}{\pi} = \underset{p=\infty}{\text{limit}}\; \frac{m}{p} = \textit{ratio of the angle to two right angles.}$$

Hence, we may properly take θ as the measure of the angle, π being taken to be that of two right angles. The unit of angle will then be the angle such that the arc of a circle subtending that angle at the center bears to the radius a ratio which, as the radius diminishes indefinitely, itself diminishes, and has unity as limit. This angle is called *a radian,* and the system of measurement of angles here adopted is called *circular measure.*

§ 38. *Unique Limits of* $\frac{\sin\theta}{\theta}$ *and* $\frac{1-\cos\theta}{\theta}$.

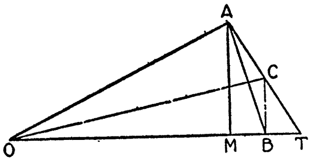


FIG. 11.

Let AB be a chord of a circle subtending a small acute angle at the center O. Let AM be the perpendicular from A on OB, AT be the tangent at A, meeting OB in T, and OC the line through O bisecting the chord AB, and meeting AT in C. Then, as we saw, the arc AB lies between the lengths of AB and $AC + CB$. But since the angles AMB and CBT are right angles,

$$AM < AB \text{ and } CB < CT;$$

therefore,

$$AM < \text{arc } AB < AT.$$

Dividing throughout by the radius, and then letting the radius diminish indefinitely,

$$\sin\theta < \theta < \tan\theta,$$

θ being the circular measure of the angle AOB. Dividing through by $\sin\theta$,

$$1 < \theta/\sin\theta < \sec\theta.$$

Letting the angle AOB decrease without limit, sec θ has the limit unity; therefore, the same is the unique limit of $\theta/\sin\theta$, or of $\sin\theta/\theta$.

Again, using the fundamental relation, to be proved as (3) in § 40,

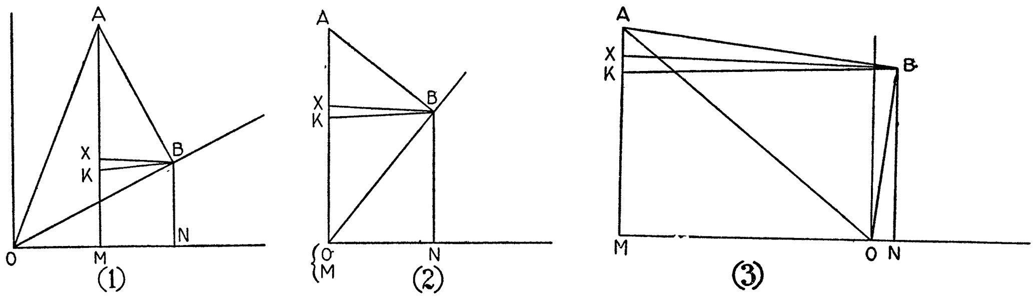
$$\frac{\theta^2}{\sin^2\theta} = \frac{\theta^2}{(1-\cos\theta)(1+\cos\theta)}$$

has the limit unity, and the second factor in the denominator has the limit 2. Hence,

$$\frac{1-\cos\theta}{\theta}$$

has the limit zero.

§§ 39–46. *Formulae for the Sine and Cosine of the Sum and Difference of Two Angles.*



(1) (2) (3)

FIGS. 12.

§ 39. Let BON and BOA be acute angles (Fig. 12 (1), (2), (3)) on opposite sides of OB, and let the angles ABO, BNO be made right angles. Also draw AM perpendicular to ON and BK to AM, and make $MX = BN$. Then we know (§ 2) that $KM < XM$, and the angle BXM is equal to the angle XBN, and is acute.

When the point A moves up to O as limiting point, so do B, K, M and N. Consequently (§ 1) the angles of the quadrilateral $BNMK$ together have four right angles for their limit, and therefore the angle KBN has a right angle for limit. The same is therefore true of the acute angle XBN. Consequently, the angle XBK has zero as limit.

For the same reason, the sum of the angles of the triangle BON has two right angles as limit, so that the sum of the angles OBN and BON has a right angle for limit. Since KBO and OBN make up KBN, which, as we saw, has a right angle for limit, it follows that KBO has BON as limit. Similarly, $KAB + ABK$ has a right angle for limit; whence KAB and KBO have the same limit, viz., as we saw, the angle BON. Now

$$AM = AK + KM = AK + XM - KX = AK + BN - KX;$$

therefore,

$$\frac{AM}{OA} = \frac{AK}{AB}\frac{AB}{OA} + \frac{BN}{OB}\frac{OB}{OA} - \frac{KX}{KB}\frac{KB}{AB}\frac{AB}{OA}.$$

Since the ratios converge uniformly to their limits, we get, letting A move up to O as limit,

$$\sin AOM = \cos BON \sin AOB + \sin BON \cos AOB. \qquad (\alpha)$$

Writing a for the angle AOB, and b for the complement of the angle BON, we get, by §§ 22 and 25,

$$\cos(a - b) = \cos a \cos b + \sin a \sin b. \tag{1}$$

Again,

$$\frac{OM}{OA} = \frac{ON - MN}{OA} = \frac{ON}{OB} \cdot \frac{OB}{OA} - \frac{MN}{BK} \cdot \frac{BK}{AB} \cdot \frac{AB}{OA}.$$

But, as before KX/KB converged uniformly to zero, and therefore KM/BN to unity, so now MN/BK has the unique limit unity. Hence, proceeding to the limit in the last equation,

$$\cos AOM = \cos BON \cos AOB - \sin BON \sin AOB, \tag{β}$$

that is, with the same notation as before,

$$\sin(a - b) = \sin a \cos b - \cos a \sin b. \tag{2}$$

§ 40. The above formulæ for $\cos(a - b)$ and $\sin(a - b)$ have been proved on the assumption that a and b are both acute. The case when $b = a$ can be considered as the limiting case of a sequence of cases in which b approaches a as limit, since the sine and cosine have been shown to be continuous. In this case also the equations must therefore hold. The sine equation then reduces to an identity, but the cosine equation gives us the fundamental relation

$$1 = \cos^2 a + \sin^2 a, \tag{3}$$

whence also, dividing by $\tan^2 a$,

$$\sec^2 a = 1 + \tan^2 a. \tag{4}$$

Thus the equations giving the sine and cosine of $(a - b)$ hold in all cases when a and b are acute angles.

§ 41. Now if in (α) and (β) we put b for the angle BON itself, instead of for its complement, we have the formulæ

$$\sin(a + b) = \sin a \cos b + \cos a \sin b, \tag{5}$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b, \tag{6}$$

provided a and b are acute.

§ 42. If we change b into $-b$ the formulæ (1) and (6) are interchanged, and so are (2) and (5); therefore we may consider (5) and (6) only, as including (1) and (2), a being a positive acute angle and b a positive or negative acute angle. If we change a into $-a$, however, the formula (5) becomes (2), and (6) becomes (1); hence, (5) and (6) hold when either a or b, or both, are positive or negative acute angles.

Again, if we add $\frac{\pi}{2}$ to a, or to b, (5) and (6) are interchanged, and are therefore still true. Since these operations together with the preceding ones may be carried out any number of times, this shows that the formulæ (5) and (6) hold for any positive or negative values of a and b, including those which are multiples of a right angle, since the formulæ clearly hold for zero values of a or b, or both.

§ 43. Putting $b = a$, we get

$$\sin 2a = 2 \sin a \cos a,$$
$$\cos 2a = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a.$$

§ 44. Hence,

$$2 \sin^2 \tfrac{1}{2} a = 1 - \cos a,$$
$$2 \sin \tfrac{1}{2} a \cos \tfrac{1}{2} a = \sin a,$$

whence

$$\tan \tfrac{1}{2} a = (1 - \cos a)/\sin a.$$

Applying this to the case when a is a right angle,

$$\tan \frac{\pi}{4} = 1, \ \sin \frac{\pi}{4} = \cos \frac{\pi}{4} = 1/\sqrt{2}.$$

§ 45. Putting $b = 2a$, we get

$$\sin 3a = 3 \sin a - 4 \sin^3 a,$$
$$\cos 3a = 4 \cos^3 a - 3 \cos a.$$

§ 46. These equations may be used to give equations for the sine and cosine of $\frac{a}{3}$. In particular, putting $3a = \pi/2$, the second equation gives, after dividing by $\cos^3 a$,

$$0 = 4 - 3\sec^2 a = 1 - 3\tan^2 a,$$

so that

$$\tan \frac{\pi}{6} = 1/\sqrt{3}, \ \sin \frac{\pi}{6} = \frac{1}{2}, \ \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

Hence, by the formulæ for the complementary angle,

$$\tan \frac{\pi}{3} = \sqrt{3}, \ \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}, \ \cos \frac{\pi}{3} = \tfrac{1}{2}.$$

§ 47. *Differential Coefficients of the Sine, Cosine and Tangent.*

From (5) we have

$$\frac{\sin(a+h) - \sin a}{h} = \sin a \frac{\cos h - 1}{h} + \cos a \frac{\sin h}{h},$$

whence, if we are using circular measure, by § 38, the left-hand side has, as h approaches zero in any manner, a unique limit, viz., $\cos a$. Thus $\sin a$ is a differentiable function and

$$\frac{d}{da}\sin a = \cos a;$$

similarly from (6),

$$\frac{d}{da}\cos a = -\sin a;$$

whence also

$$\frac{d}{da}\tan a = \sec^2 a.$$

Hence, it follows that *the sine, cosine and tangent possess all their differential coefficients of every order.*

§§ 48–51. *Continuity and Differentiability of the Length $2\pi f(r)$ of a Circle of Radius r.*

§ 48. The length of a circle of radius r is evidently a function of r alone; let it be $2\pi f(r)$. Then, by what precedes, $f(r)$ has a unique limit, when r is indefinitely decreased, and this limit is unity. It is almost immediately evident that $f(r)$ is a continuous function of r. For the length of the circle is the limit of the monotone ascending sequence of inscribed regular polygons, and also of the monotone descending sequence of circumscribed regular polygons. From §§ 7 and 12, where it was shown that the sine-ratio and the cosine-ratio are continuous functions, it follows that the tangent-ratio — viz., that of the perpendicular to the base — is so also, and hence it follows that the perimeters of the circumscribed polygons are continuous functions of the radius. Thus the length of the circle, being the limit both of a monotone ascending and of a monotone descending sequence of continuous functions, is itself a continuous function of the radius.* Hence, *$f(r)$ is a continuous function of r.*

§ 49. Using the notation of § 3 and Fig. 2, we had

$$CP - BN < AM - CP,$$

whence, supposing these lengths to be the halves of corresponding sides of regular polygons, with O as common center, whose perimeters are p_B, p_C, p_A,

$$p_C - p_B < p_A - p_C.$$

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* W. H. Young: "On a Test for Continuity," 1908, *Proc. Royal Society of Edinburgh*, Vol. XXVIII, pp. 217–221.

Since this is true for all such polygons, we have, denoting the circumference of the circle through A by c_A, and so for other points,

$$c_C - c_B < c_A - c_C.$$

Now suppose that we have $(n+m)$ equal stretches marked off along the radius, and we have the corresponding inequalities,

$$\begin{gathered} c_3 - c_2 > c_2 - c_1, \\ c_4 - c_3 > c_3 - c_2, \\ \dots\dots\dots\dots\dots, \\ c_{n+m+1} - c_{n+m} > c_{n+m} - c_{n+m-1}. \end{gathered}$$

Hence, it follows that

$$c_{n+m+1} - c_{n+1} > c_{n+m} - c_n > c_{n+m-1} - c_{n+m-m} > \dots > c_{m+1} - c_1.$$

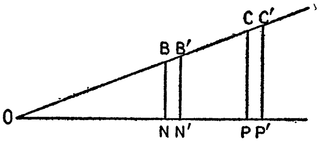

FIG. 13.

Thus if B' lie on the other side of B from O, and $BC = B'C'$, then if BC/CC' is rational, and is in the ratio n/m, we have

$$c_{C'} - c_{B'} > c_C - c_B.$$

Now if BC be fixed and $= h$, the points whose distances from O are rational with respect to h are dense everywhere, and the above shows that the function

$$\frac{f(r+h) - f(r)}{h}$$

is a monotone increasing function of r, with respect to this everywhere-dense set of points. But it is also a continuous function of r (§ 48); therefore it is a monotone increasing continuous function of r.

§ 50. Hence, if, as before, $BC = CA$,

$$\frac{f(OC) - f(OB)}{BC} < \frac{f(OA) - f(OC)}{CA},$$

adding numerators and denominators, we see that the first fraction is

$$< \frac{f(OA) - f(OB)}{OB}.$$

That is,

$$\frac{f(r+h) - f(r)}{h} < \frac{f(r+2h) - f(r)}{2h}.$$

Hence, by the same reasoning as before, $\frac{f(r+h)-f(r)}{h}$ is a monotone increasing continuous function of h, r being constant.

§ 51. From the preceding result it follows that $f(r)$ has a right-hand differential coefficient for every positive value of r. But the incrementary ratio $\frac{f(r+h)-f(r)}{h}$ has been shown to be a monotone increasing function of r; therefore, the same is true of its unique limit, viz., the right-hand differential coefficient. Therefore, this right-hand differential coefficient has a unique limit on each side, and that unique limit is therefore its value there, by a known property of derivates, and is the common value of all the derivates there, and the common value of their limits on the two sides. Thus $f(r)$ *possesses a differential coefficient, and it is a monotone increasing continuous function.*

§§ 52–55. *The Defect of a Triangle.*

§ 52. Let ABC be a right-angled triangle, with the angle C a right angle. Divide BC into four equal parts at C_2, C_1 and D_2 (Fig. 14). Let A_2C_2 and A_1C_1 be perpendicular to BC and meet BA in A_2 and A_1 respectively. Also draw the perpendicular D_2Q_2 from D_2 to BC and make it of length equal to C_2A_2. Join BQ_2 and produce it to meet AC in P_2, which it must do, since the point Q_2 obviously lies inside the triangle.

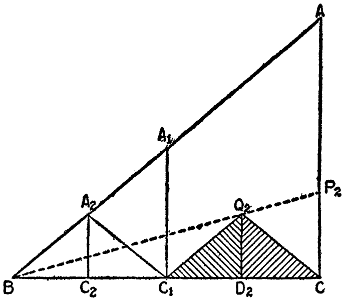

FIG. 14.

Now the triangles A_2BC_2, $A_2C_2C_1$, $Q_2C_1D_2$, Q_2D_2C are all congruent, and therefore have the same *defect*; that is, the difference between two right angles and the sum of the angles of the triangle. Let this defect be d_2, and the defect of the triangle P_2BC be d_2'. Then, since the two triangles with vertex Q_2 are internal to the triangle P_2BC,

$$2d_2 < d_2'. \tag{1}$$

Also

$$\tan P_2BC < Q_2D_2/BD_2 < A_2C_2/3BC_2. \tag{2}$$

Let us next divide into eight, then into sixteen, ... equal parts, and at each stage let us form triangles on the segments between the points of division, congruent to the first such triangle, viz., A_nBC_n, where A_nC_n is perpendicular to BC, and A_n lies on BA. Let Q_n be the vertex of the first of these triangles on the other side of the middle point C_1 from B, so that there are 2^{n-1} of the triangles between C_1 and C. Join BQ_n and produce to meet AC in P_n. Then, denoting the defects of the triangles A_nBC_n and P_nBC by d_n and d'_n, we have two relations at each stage corresponding to (1) and (2); viz.,

$$2^{n-1}d_n < d'_n,$$

$$\tan P_nBC < A_nC_n/(2^{n-1}+1)BC_n.$$

Proceeding to the limit with the latter relation, we see that $\tan P_nBC$ converges to zero, since A_nC_n/BC_n converges to $\tan ABC$. Hence, the triangle P_nBC folds up into the straight line BC, so that its defect d'_n has the unique limit zero. Hence, by the former of the two relations, $2^{n-1}d_n$ has the unique limit zero, and therefore so has $2^n d_n$. But

$$2^nBC_n = BC;$$

therefore

$$d_n/BC_n \text{ has the unique limit zero.}$$

§ 53. Now let C'' be any point between C_n and C_{n-1}. Draw $C''A''$ perpendicular to BC, meeting BA in A'', and let d'' be the defect of the triangle $A''BC''$. Then, since this triangle lies inside the triangle $A_{n-1}BC_{n-1}$,

$$d'' < d_{n-1},$$

and since C'' lies between C_n and C_{n-1},

$$BC'' > \tfrac{1}{2}BC_{n-1}.$$

Hence,

$$d''/BC'' < \tfrac{1}{2}d_{n-1}/BC_{n-1}.$$

Hence, d''/BC'' has also the unique limit zero as C'' approaches B as limiting point in any manner.

§ 54. In the above discussion the angle ABC was kept fixed. Now let us take any sequence of right-angled triangles, $\Delta_1, \Delta_2, \ldots.$, such that one of the sides $x_1, x_2, \ldots.$ containing the right angle decreases with zero as unique limit. Then the acute angle adjacent to x_n has a definite upper bound, less than or equal to a right angle. If this upper bound is less than a right angle, we can draw a fixed angle ABC equal to it, and fit the successive triangles $\Delta_1, \Delta_2, \ldots.$ into this angle, as for instance $D''BC''$ in Fig. 15. If $D''C''$ meet AB in A'', the defect of $D''C''B$ is less than that of $A''C''B$, and therefore this defect divided by BC'' has zero as unique limit as C'' approaches B as limiting point.

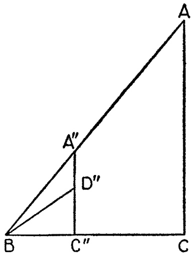

FIG. 15.

If, however, the upper bound is a right angle, it must be the upper limit. Thus if we choose out a sub-sequence of the triangles having other than a right angle as the unique limit of the acute angle in question, the preceding investigation holds. If, however, this acute angle has a right angle as unique limit, the second of the acute angles has the unique limit zero, and has therefore an upper bound which is less than a right angle. Thus, if we know that the hypotenuse, and therefore also the other side containing the right angle, has the limit zero, it follows from the preceding that the limit of the ratio of the defect to that second side, *a fortiori* to the hypotenuse, is zero. Thus in any case, whether or no the angle first in question has the upper limit a right angle, the limit of the defect over the hypotenuse is zero.*

Taking the hypotenuse to be of the first order of small quantities, we may therefore say that the defect of a right-angled triangle is of order higher than the first.

§ 55. Since any triangle may be divided into two right-angled triangles, it follows that *if all of the sides of the triangle be considered to be of the first order of small quantities, the defect of the triangle is of order higher than the first.*

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* This is clearly true whether or no the hypotenuse has zero as limit. For, if not, the triangle folds up into a straight line and therefore the defect vanishes.

§ 56. *The Approximate Sine Formulae.*

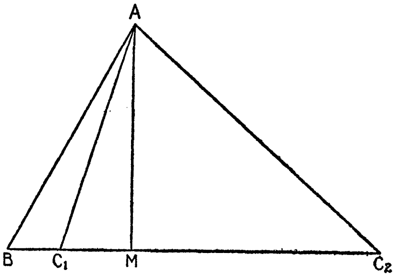

FIG. 16.

Let ABM be a triangle with a right angle at M. Then, since the defect is of order higher than the first,

$$A\hat{B}M = \frac{\pi}{2} - B\hat{A}M - e_2,$$

or, say,

$$B = \frac{\pi}{2} - A - e_2.$$

Hence,

$$\sin B = \cos(A + e_2) = \cos A \cos e_2 - \sin A \sin e_2 = \cos A - e',$$

where, by § 38, e'_2, like e_2, is of order higher than the first.

But, as proved in §§ 9 and 12,

$$\sin B < AM/AB < \cos A.$$

Therefore,

$$0 < \frac{AM}{AB} - \sin B < e'_2.$$

Hence, *AM differs from $AB \sin B$, and similarly from $AB \cos A$, by a quantity which is of an order higher than the second of small quantities.*

Therefore, if ABC is any triangle, whose angles at B and C are neither of them a right angle, and we drop the perpendicular AM on CB, AM differs by a quantity of order higher than the second from each of the expressions

$$c \sin B \text{ and } b \sin C,$$

a, b and c denoting as usual the sides opposite the angles A, B and C respectively. Hence, *neglecting small quantities of order higher than the second,*

$$\frac{\sin B}{b} = \frac{\sin C}{c},$$

and similarly $= \sin A/a$, it being noticed that, by the above, these approximate equations hold whether or no one of the angles A, B and C is a right angle, in which case the sine is unity.

§ 57. *Approximate Equality of the Arc and Its Chord.*

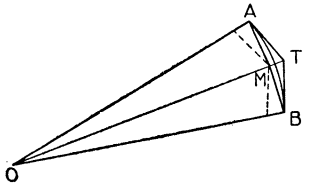

FIG. 17.

Let A and B be neighboring points on a circle of center O, M the middle point of the chord AB, and TA, TB the tangents at A and B, so that OMT is a straight line, and the angles at M are right angles.

Then we know that the arc AB is less than $AT + TB$; therefore, the half arc is less than AT. Hence, also, the half arc of the circle with OM as radius is less than AM. It follows that

$$\text{arc } AB - \text{chord } AB < \text{arc } AB - \text{arc with } AM \text{ as radius}$$
$$< (f(OA) - f(OM))\, A\hat{O}B < \frac{f(OA) - f(OM)}{OA - OM}\, TM \,.\, A\hat{O}B.$$

Now the incrementary ratio in the last expression has $f'(AO)$ as unique limit, and is therefore finite; $A\hat{O}B$ is of the first order, and TM of order higher than the first, since TM/AM converges uniformly to tan 0, that is, to zero. Thus *the difference between the chord and the arc is of order higher than the second.*

§§ 58, 59. *Killing's Formulae.*

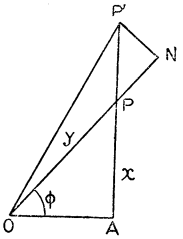

FIG. 18.

§ 58. Let P' be a point near P on a fixed straight line AP (Fig. 18), so that PP' and the angle POP', or $\Delta\phi$, are to be considered as of the first order of small quantities. Let OA be the initial line from which the angle ϕ, that is AOP, is measured. Draw $P'N$ perpendicular to OP, and let us denote

$$\begin{aligned} &OP \text{ by } y, \quad OP' \text{ by } y+\Delta y, \\ &AP \text{ by } x, \quad AP' \text{ by } x+\Delta x. \end{aligned}$$

Then, by § 54, $P'N$ differs from $f(y+\Delta y)\,\Delta\phi$ by a small quantity e_{2+} of order higher than the second. Also, as P' moves up towards P, $P'N/PP'$ converges to $\sin\psi$, where ψ is the angle OPA. Hence, $\dfrac{\Delta x}{\Delta\phi}$ converges to $f(y)\operatorname{cosec}\psi$. In other words, x has a differential coefficient with respect to ϕ, and

$$\frac{dx}{d\phi} = f(y)\operatorname{cosec}\psi.$$

Again,

$$\Delta y = OP' - OP = OP' - ON + PN;$$

therefore, since, by § 54, ON differs from $OP'\cos P'ON$ by a small quantity e'_{2+} of order higher than the second,

$$\frac{\Delta y}{\Delta x} = (y+\Delta y)\,\frac{1-\cos\Delta\phi}{\Delta\phi}\,\frac{\Delta\phi}{\Delta x} + \frac{PN}{PP'} + e'_{2+}.$$

The first term here converges to zero, since, by the above, the third as well as the first factor remains bounded, and, by § 38, the second factor converges to zero. Hence, the left-hand side, like the right-hand side, converges to $\cos\psi$. That is, y possesses a differential coefficient with respect to x, and

$$\frac{dy}{dx} = \cos\psi.$$

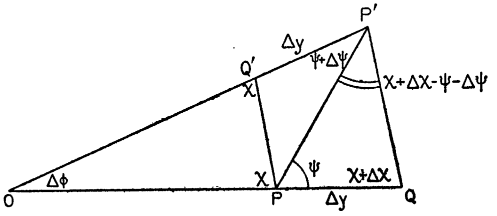

FIG. 19.

§ 59. Now on OP and OP' respectively take Q and Q', so that

$$PQ = Q'P' = \Delta y.$$

Let the equal angles OPQ' and $OQ'P$ be denoted by χ, and OQP' and $OP'Q$

by $\chi + \Delta\chi$. Then, neglecting small quantities of order higher than the first, we get, by considering the triangles $Q'PP'$ and QPP' (Fig. 19),

$$(\pi - \chi) + (\pi - \psi - \chi) + (\psi + \Delta\psi) = \pi,$$
$$\psi + (\chi + \Delta\chi) + (\chi + \Delta\chi - \psi - \Delta\psi) = \pi.$$

Hence,

$$\chi = \frac{\pi}{2} + \tfrac{1}{2}\Delta\psi,$$

$$\chi + \Delta\chi = \frac{\pi}{2} + \tfrac{1}{2}\Delta\psi,$$

and therefore,

$$\pi - \chi - \psi = \chi + \Delta\chi - \psi - \Delta\psi = \frac{\pi}{2} - \psi - \tfrac{1}{2}\Delta\psi;$$

that is,

$$\angle\, Q'PP' = \angle\, PP'Q = \frac{\pi}{2} - \psi - \tfrac{1}{2}\Delta\psi.$$

Therefore, projecting first $Q'P$ and PQ, and then $Q'P'$ and $P'Q$ perpendicular to PP', and equating the results, we get, neglecting small quantities of order higher than the second, by § 56,

$$PQ' \sin Q'PP' + PQ \sin\psi = P'Q \sin PP'Q + Q'P' \sin(\psi + \Delta\psi),$$

or, remembering that we may replace the chord by the arc, if we are neglecting small quantities of order higher than the second (§ 57),

$$f(y)\,\Delta\phi \cos(\psi + \tfrac{1}{2}\Delta\psi) + \Delta y \sin\psi = f(y + \Delta y)\,\Delta\phi \cos(\psi + \tfrac{1}{2}\Delta\psi) + \Delta y \sin(\psi + \Delta\psi).$$

Hence, rearranging terms and dividing through by $\Delta y\,\Delta\phi$, so that the neglected terms, which we now re-introduce, contribute a term whose limit is zero,

$$\frac{f(y + \Delta y) - f(y)}{\Delta y} \cdot \cos(\psi + \tfrac{1}{2}\Delta\psi) = -\frac{\sin(\psi + \Delta\psi) - \sin\psi}{\Delta\psi} \cdot \frac{\Delta\psi}{\Delta\phi} + e' = \cos\psi\frac{\Delta\psi}{\Delta\phi} + e,$$

where e' and e have the limit zero.

Since every term in this equation except $\Delta\psi/\Delta\phi$ is known to have a unique limit, when P' moves up to P, it follows that that ratio also has a unique limit, so that ψ has a differential coefficient with respect to ϕ, given by

$$\frac{d\psi}{d\phi} = -f'(y),$$

provided $\cos\psi$ is not zero.

§ 60. *The Sine Formulae.*

We have now fully justified the following equations, obtained first by Killing. In this and the following articles we follow that author, in obtaining, for the convenience of readers, the fundamental formulæ connecting the sides and angles of any triangle.

$$\left.\begin{aligned}\frac{dx}{d\phi} &= f(y)\operatorname{cosec}\psi,\\ \frac{dy}{dx} &= \cos\psi,\\ \frac{d\psi}{d\phi} &= -f'(y),\end{aligned}\right\}\ \text{(Killing's equations).}$$

In other words, assuming that the length OA and the angle OAP are kept constant, we have the ratios of the increments of the remaining four parts of the triangle POA. Thus, in particular,

$$\frac{dy}{d\psi} = -\frac{\cos\psi}{\sin\psi}\cdot\frac{f(y)}{f'(y)},$$

integrating which, we get

$$f(y)\sin\psi = \text{const.}$$

Since as P moves up to A, x approaches zero, y approaches OA, or say c, and ψ approaches $\pi-\beta$, where β is the angle OAP, we have

$$f(y)\sin\psi = f(c)\sin\beta.$$

Hence, in any triangle ABC, with the usual notation for the sides and angles,

$$\frac{f(a)}{\sin A} = \frac{f(b)}{\sin B} = \frac{f(c)}{\sin C}. \tag{I}$$

§ 61. *The Cosine Formulae.*

Thus, returning to our previous notation,

$$f(c)\sin\phi = f(x)\sin\psi,$$

whence, differentiating,

$$f(c)\cos\phi\, d\phi = f'(x)\,dx\sin\psi + f(x)\cos\psi\, d\psi,$$

or, replacing the increments by their proportionals, by § 60,

$$f(c)\cos\phi = f'(x)f(y) - f(x)f'(y)\cos\psi.$$

Hence, in any triangle,

$$f(c)\cos A = f'(a)f(b) - f(a)f'(b)\cos C. \tag{II}$$

§§ 62, 63. *Conclusion. The Sine and Cosine Formulae in the Three Geometries.*

§ 62. Eliminating $\cos C$ between the last equation and the similar one,

$$f(a)\cos C = f'(c)\,f(b) - f(c)\,f'(b)\cos A,$$

we get

$$f(c)\cos A = f'(a)\,f(b) - f'(b)\,f'(c)\,f(b) + f(c)\,(f'(b))^2\cos A.$$

Therefore,

$$f(c)\cos A\,\{1-(f'(b))^2\} = f(b)\,\{f'(a) - f'(b)\,f'(c)\}.$$

Hence, interchanging b and c, and eliminating $\cos A$,

$$\frac{1-[f'(b)]^2}{[f(b)]^2} = \frac{1-[f'(c)]^2}{[f(c)]^2}.$$

Hence, since b and c are any two stretches which form two of the sides of a triangle, we have

$$\frac{1-[f'(b)]^2}{[f(b)]^2} = \text{const.} = \frac{1}{k^2}, \text{ say,}$$

for all values of b.

Integrating and remembering that $\underset{b=0}{\text{limit}}\, f(b) = 0$, we have three cases to distinguish:

1) k^2 is positive,

$$\sin^{-1}\frac{f'(b)}{k} = \frac{b}{k},$$

whence

$$f(b) = k\sin\frac{b}{k},$$

$$f'(b) = \cos\frac{b}{k}.$$

2) k^2 is negative,

$$\sinh^{-1}\frac{f'(b)}{k} = \frac{b}{k},$$

$$f(b) = k\sinh\frac{b}{k},$$

$$f'(b) = \cosh\frac{b}{k}.$$

3) k is infinite,

$$f(b) = b.$$

Now in the Lobatchefsky geometry, as we have seen, $f'(b)$ is a monotone increasing function of b; thus case 1) can only be the Riemann geometry, in which $f'(b)$ is a monotone decreasing function of b, for small values of b; case 3), in which $f'(b)$ is constant, is the Euclidian case, and case 2) is the Lobatchefsky geometry.

§ 63. Thus in the Lobatchefsky geometry we have:

$$\frac{\sin A}{\sinh \frac{a}{k}} = \frac{\sin B}{\sinh \frac{b}{k}} = \frac{\sin C}{\sinh \frac{c}{k}}, \qquad \text{(Lobatchefsky I)}$$

$$\sinh \frac{c}{k} \cos A = \cosh \frac{a}{k} \sinh \frac{b}{k} - \sinh \frac{a}{k} \cosh \frac{b}{k} \cos C. \qquad \text{(Lobatchefsky II)}$$

In the Euclidian geometry we have:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \qquad \text{(Euclid I)}$$

$$c \cos A = b - a \cos C. \qquad \text{(Euclid II)}$$

Finally, in the Riemann geometry, we have:

$$\frac{\sin A}{\sin \frac{a}{k}} = \frac{\sin B}{\sin \frac{b}{k}} = \frac{\sin B}{\sin \frac{c}{k}}, \qquad \text{(Riemann I)}$$

$$\sin \frac{c}{k} \cos A = \cos \frac{a}{k} \sin \frac{b}{k} - \sin \frac{a}{k} \cos \frac{b}{k} \cos C. \qquad \text{(Riemann II)}$$

From these formulæ all the others can be obtained.

GENEVA, *April*, 1910.